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Topological proof for the Alexander-Orbach conjecture

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This paper advocates an unconventional analytical approach to studying the fractal geometry of percolation at the threshold, which is based on the most general methods of the differential topology. Our particular interest concentrates on the Alexander-Orbach (AO) conjecture [J. Phys. (France) Lett. **43**, L625 (1982)], which assigns the universal (mean-field) value 4/3 to the spectral fractal dimension $\tilde{d}$ at the percolation threshold for all embedding Euclidean dimensions n greater than one, i.e., $n \geqslant 2$. Using the topological arguments, we show that the AO conjecture might be improved for the relatively low embedding dimensions $2 \leqslant n \leqslant 5$, for which the analytical result $\tilde{d} = 1.327 \pm 0.001$ is proposed, instead of the original AO estimate 4/3. Meanwhile we assume that the exact value $\tilde{d} = 4/3$ holds for all $n \geqslant 6$, as it follows directly from the well-known mean-field theory. The improved value of $\tilde{d} \approx 1.327$ for $2 \leqslant n \leqslant 5$ is obtained from an analysis of the basic *topological* properties of the percolating fractal sets at the threshold of percolation. We show that these properties could be investigated fruitfully with the introduction of the concept of the *fractal manifold*, which might serve as an effective instrument when considering the topology of the fractal objects. Our results indicate that the proposed value of $\tilde{d} \approx 1.327$ for the spectral dimension at the percolation threshold has the fundamental topological origin related to the most general features of the fractal geometry of percolation at criticality. We argue that the constraint $2 \leqslant n \leqslant 5$ on the topological dimension n of the embedding Euclidean space is the direct consequence of the famous Whitney theorem, which establishes the embedding properties of manifolds from the viewpoint of their dimensionality. A simple topological condition that identifies the threshold of percolation is obtained for $2 \leqslant n \leqslant 5$. The particular topological restrictions implied throughout the present study are discussed and the important issue of contractibility of the fractal manifolds is pointed out. [S1063-651X(97)02308-8]

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I. INTRODUCTION

Applications of percolation theory (see, e.g., Refs. [1,2]) have led to remarkable advances in the understanding of many phenomena related to the formation of irregular structures. Topological properties of irregular, random configurations recently have received a good deal of attention in association with the possible universal nature of the geometry of percolation in the vicinity of the so-called percolation threshold [3,4].

Indeed, consider an infinite, statistically homogeneous, isotropic random scalar field $\psi(\mathbf{x})$, where $\mathbf{x}$ is an n-dimensional Euclidean vector $(n \geqslant 2)$. The introduction of an arbitrary threshold h makes it possible to divide the space into two topologically different parts: one composed of all regions where $\psi(\mathbf{x}) < h$, marked as being ''empty,'' and the other composed of the regions where $\psi(\mathbf{x}) > h$, marked as being ''filled.'' One of these parts will include a connected infinite set, which is said to ''percolate.'' Changing the threshold h, one can find the critical threshold h_c when the topological phase transition happens (i.e., the nonpercolating part starts to percolate or vice versa).

It has been recognized [1,2,4,5] that the geometry of the percolating set at criticality is a typical fractal and that the anomalous behavior [6] of the macroscopic quantities at the point of the percolation transition is due to the divergence of the percolation correlation length ξ. At length scales χ large compared to ξ, the geometry of the field $\psi(\mathbf{x})$ looks, in the statistical sense, homogeneous, so that the *macroscopic* average density of the field $\psi(\mathbf{x})$ is constant. However, comprehensive numerical studies indicate (see, e.g., Ref. [2]) that the percolating sets are not homogeneous at length scales χ, which are in the range $a \ll \chi \ll \xi$, the quantity a being the microscopic ''lattice'' distance. (At the percolation threshold, $\xi/a \to \infty$.) In this range, the sets are *self-similar* and their averaged density scales with χ as $\propto \chi^{D-n}$. Here D is the fractal dimension of the set, commonly referred to as the Hausdorff dimension of the fractal [7], and n is the topological (integer) dimension of the embedding Euclidean space, which is always not less than D, i.e., $D \leqslant n$.

The fractal dimension D is not, however, the only geometric parameter required for the complete description of self-similar fractal objects. The other is the index of connectivity θ [4,6,8–10]; contrary to the fractal dimension D that describes the scaling behavior of the averaged ''mass'' density of a fractal set, the index θ quantifies how the ''elementary'' structural units (each of size approximately equal to a) inside the set (e.g., the filled and empty sites for the problem of the site percolation on lattices) are ''glued'' together to form the entire fractal object. (Roughly speaking, the parameter θ describes the ''shape'' of a fractal set and may be different for fractals even with equal values of the fractal dimension D. The geometric sense of the index θ is discussed on a descriptive level in Ref. [11].) It is worth mentioning that the index of connectivity θ plays an important role in many dynamical phenomena on fractals, e.g., transport processes in disordered media [6,8–10,12,13], ''bimolecular'' chemical reactions [14,15], and localization of waves [4,16,17]. The original important promotion of the parameter θ was given in the pioneering paper [6], where the concept of the range-dependent diffusion on percolating networks was proposed. By applying the scaling theory, it has been shown [6] that the diffusion constant on a percolating network, for length scales χ between a and ξ, behaves as a power law proportional to $\chi^{-\theta}$.

This insight, along with the realization that solving the problem of the range-dependent diffusion was equivalent to solving the (scalar) elastic vibration problem (for more details see, e.g., Ref. [4]), led Alexander and Orbach [3] to evaluate the density of states for vibrations of a percolating network at criticality (these vibrations were termed fractons), with the introduction of the so-called fracton, or spectral, dimension $\tilde{d}$. This new quantity was defined as a specific combination of the fractal dimension D and the index of connectivity θ and has the form $\tilde{d}\equiv 2D/(2+\theta)\leqslant D$. In addition, Alexander and Orbach [3] noted that the spectral dimension $\tilde{d}$ for the percolating networks at criticality was numerically remarkably close to the mean-field value 4/3 (exact in Euclidean dimension $n=6$) for all embedding dimensions n greater than one, even though the parameters D and θ were by no means constant as functions of n (below $n=6$). This numerical evidence led them to speculate that the spectral dimension $\tilde{d}$ might be exactly 4/3 for the percolating networks at criticality in all embedding dimensions $n\geqslant 2$. This has come to be known as the Alexander-Orbach (AO) conjecture.

Much theoretical and numerical effort has been made in the attempt to prove or disprove the AO conjecture (for a comprehensive review see, e.g., Refs. [4,10]). This conjecture is important because, if true, it would allow one to describe the fractal geometry of percolation by using the unique basic concept of the spectral dimension $\tilde{d}=4/3$ for such fundamental problems as the correlated and uncorrelated percolation on lattices as well as for the more general continuum percolation problem [2,4]. The great interest in this conjecture results not only from the universal value 4/3 assigned to $\tilde{d}$, but also from the fact that it establishes a relationship between the index of connectivity θ that appears in the description of the dynamical processes on fractals and the Hausdorff fractal dimension D, yielding the scaling behavior of the density of the fractal substrate.

At present, there exists a rigorous mathematical treatment of the topology of percolation for the sufficiently high embedding dimensions $n\geqslant 6$, which is based on the mean-field theory. The mean-field percolation can be modeled by the percolation on a Cayley tree (Bethe lattice). A Cayley tree is defined as a graph without loops in which each node has the same number of branches; the self-similarity of such graphs is not necessarily manifest in their geometric representation, but is seen in their connectivity [18]. The percolation problem on Cayley trees has been solved exactly by Coniglio [19] and the rigorous analytical result $\tilde{d}=4/3$ has been established for all $n\geqslant 6$ (see also the review by Havlin and Ben-Avraham [10]). This result proves the AO conjecture in $n\geqslant 6$.

For the *lower* embedding dimensions $2\leqslant n\leqslant 5$, the mean-field theory cannot be directly applied and the analytical consideration of the topology of percolation in these dimensions meets considerable difficulties [10]. Meanwhile, a large body of studies, both theoretical and numerical, indicates that the true value of the spectral dimension $\tilde{d}$ is slightly *smaller* than 4/3 for $2\leqslant n\leqslant 5$ (for a review see, e.g., Ref. [4]). (This smaller value might be a remarkably accurate estimate of $\tilde{d}$ for percolation in all $2\leqslant n\leqslant 5$ being, nevertheless, quite close, although not equal, to 4/3.) However, no rigorous analytical proof or disproof of the AO conjecture has been obtained for $2\leqslant n\leqslant 5$ and the validity of this conjecture for such embedding dimensions n still remains an open question [4,10].

The goal of the present study is to show *analytically* that the AO conjecture might be *valid* for percolation in embedding Euclidean dimensions $2\leqslant n\leqslant 5$ if a slight improvement of the original AO result 4/3 is made for these n. Indeed, we intend to prove that the spectral fractal dimension $\tilde{d}$ for the percolating networks at criticality might be assigned the universal value approximately equal to 1.327 in all embedding dimensions $2\leqslant n\leqslant 5$; we note that this value is indeed *smaller* than the classical (mean-field) result 4/3, i.e.,

$$\tilde{d}\equiv 2D/(2+\theta)\approx 1.327<4/3, \quad 2\leqslant n\leqslant 5. \tag{1}$$

Thus we are considering Eq. (1) as the possible improved form of the AO conjecture for $2\leqslant n\leqslant 5$.

To give support for the improved AO conjecture (1), we propose below an unconvential analytical treatment of the fractal geometry of percolation, which is based on the most general *topological* methods. (Some topological aspects of percolation in random scalar fields have been recently discussed in Ref. [20].) The basic concept of our study will be that of the *fractal manifold* to be introduced in Sec. II. We show that this concept might be an effective analytical tool when dealing with the basic geometric features of the fractal objects, which yields a key to the description of fractals in the framework of the *differential topology*. In Sec. II we exploit the concept of the fractal manifold to quantify the topological properties of the percolating fractal sets at criticality. This approach is then used in Secs. III and IV for the direct analytical calculation of the spectral fractal dimension $\tilde{d}$ at the percolation threshold [see Eq. (1)]. It is proven in these sections that the proposed estimate (1) has the fundamental *topological* origin related to the intrinsic geometric properties of the percolating sets at criticality. The relevant topological constraints on the embedding dimension n are discussed in detail in Sec. V. With use of the famous *Whitney theorem* on the embedding of manifolds [21], we show in Sec. V that the improved form (1) of the AO conjecture could be indeed valid only for the relatively low embedding dimensions n, which lie in the above range $2\leqslant n\leqslant 5$. We now turn to the consideration of the general topological proof (as rigorous as possible under the limitations of an article) for the AO conjecture in the form (1).

II. BASIC TOPOLOGICAL CONCEPTS

We start with an infinite, statistically homogeneous, isotropic random scalar field $\psi(\mathbf{x})$, where $\mathbf{x}\in E^n$ and $n\geqslant 2$. (Here E^n denotes n-dimensional Euclidean space. The parameter n is defined to be integer.) Assume the condition $|\nabla\psi(\mathbf{x})|\neq 0$ everywhere, except perhaps points of negligible measure. [This condition implies that the field $\psi(\mathbf{x})$ is nondegenerated for almost all $\mathbf{x}$. Note that the infinite values of $\nabla\psi(\mathbf{x})$, i.e., when $|\nabla\psi(\mathbf{x})|=+\infty$, are allowed.]

Let $h=h_c$ be the critical percolation threshold; this divides E^n into the two topologically distinct parts: $\psi(\mathbf{x})<h_c$ and $\psi(\mathbf{x})>h_c$. Without loss of generality, we may assume it is the part $\psi(\mathbf{x})<h_c$ that includes the percolating (i.e., connected infinite) set, which we shall denote by F_c. Because the topology of percolation at criticality is associated with the fractal behavior [1,2], we consider the set of points F_c as a self-similar fractal object with the Hausdorff dimension D and the spectral dimension $\tilde{d}=2D/(2+\theta)$. We also introduce the standard notation ∂F_c for the boundary of the set F_c. It is clear that whereas F_c is defined by the inequality $\psi(\mathbf{x})<h_c$, the boundary ∂F_c is determined by the equation $\psi(\mathbf{x})=h_c$.

Let us remark that we use the term ''connected'' in the precise topological sense ''path connected.'' ''Path connected'' means that along with any its two points, the path-connected set also includes a ''path'' that goes from one of these points to another [22]. A path in F_c is defined as a continuous mapping $\phi\colon \bar{I}\rightarrow F_c$ of the closed unit interval $\bar{I}\equiv[0,1]$ into F_c. The points $\phi(0)$ and $\phi(1)$ represent, by definition, the initial and the final points of the path. A path in F_c can be viewed geometrically as a continuous spatial curve γ_ϕ that goes from $\phi(0)$ to $\phi(1)$ and represents the set of all points $\phi(\bar{I})$. Note that the mapping ϕ that appears in the definition of the term ''path'' is not *one to one* and *mutually* continuous in general [22], so that the curve γ_ϕ may have an arbitrary number of points of self-intersections. Note also that due to the path connectedness of the set F_c, the spectral fractal dimension $\tilde{d}\geqslant 1$. Our goal now is to calculate the spectral dimension $\tilde{d}$ for the set F_c, assuming the fractal topology of percolation at the threshold.

To perform such a calculation independently of the particular value of the embedding dimension n, we propose an untraditional approach based on the formal introduction of a coordinate system on a fractal object. Following Ref. [23], we first supply the fractal set F_c with the specific topological structure of the *fractal manifold*. This means that we represent the set F_c as a union of a finite or denumerable number of domains, each topologically equivalent to a domain of k-dimensional Euclidean space E^k, where the parameter k will be quantified below. The local coordinate systems $x_1,\ldots,x_k$ for each domain can be then defined as the standard Euclidean coordinates in E^k, whereas the transition from one local coordinate system to another is quantified by the so-called matrix of the transition functions, which explicitly describes the global representation of F_c as the union of the local Euclidean domains [21,24]. (The idea to define a local coordinate system in F_c is therefore reduced to ''mapping'' of the Euclidean coordinate basis introduced in the Euclidean space E^k into the set F_c. The ''gluing'' of different *local* coordinate systems by means of the matrix of the transition functions [21,24] determines the particular *global* topological structure of the fractal manifold.) Thus we consider the fractal manifold as the topological object that is *locally* identical to the Euclidean space E^k of some dimensionality k. The *global* topology of the fractal manifold is then defined by the transition functions for each pair of local coordinate systems.

The introduced concept of the *fractal* manifold basically follows the widely known definition of the *smooth* k-dimensional manifold (k=positive integer) [21,24], this being the fundamental concept of the standard differential geometry. The only formal difference between the above two concepts is that the parameter k for the *fractal* manifold may not be necessarily the positive integer number, but is naturally allowed to take arbitrary fractional values. Below we refer to the parameter k as the dimensionality of the fractal manifold. [Strictly speaking, to give the rigorous definition of the fractal manifold, one should also generalize the notion of the derivatives of the transition functions in the light of the fractional differential calculus [5]; this might be important when establishing the topological equivalence between the manifolds having the *fractional* values of the dimensionality k. Although such a complicated discussion mostly will be beyond the scope of the present article, we draw attention to the following important point. Namely, throughout the present study, the concept of the ''topological equivalence'' implies, by definition, that the *fractal* geometries of any two topologically equivalent fractal manifolds are identical, i.e., are characterized by the same geometric parameters, so that the dimensionalities k of the topologically equivalent fractal manifolds are always assumed to coincide. This in turn implies that the derivatives of the transition functions must remain continuous up to the sufficiently high, perhaps fractional [5] order consistent with the given value of k. The continuity of all the derivatives of the transition functions up to any required (fractional) order is therefore presupposed below when necessary.]

The explicit numerical value of k is determined by the fractal geometry of the set F_c. Indeed, it can be shown [23] that the dimensionality k actually coincides with the spectral fractal dimension of the set $\tilde{d}$, i.e., $k\equiv\tilde{d}$. Consequently, we determine the topology of the set F_c as that of the fractal manifold that is *locally* identical (i.e., topologically equivalent) to the Euclidean space $E^{\tilde{d}}$ of the (fractional) dimensionality $\tilde{d}$ equal to the spectral fractal dimension of the set F_c. (The relation $k\equiv\tilde{d}$ could be illustrated by saying that the spectral fractal dimension $\tilde{d}$ yields the ''number of degrees of freedom'' for a point particle on the fractal manifold of the dimensionality $k=\tilde{d}$.)

We now need to explain how the space $E^{\tilde{d}}$ with the fractional value of the dimensionality $\tilde{d}$ could be introduced. Assume first that the parameter $\tilde{d}$ is a positive integer number and let I denote the (open) unit interval $I\equiv(0,1)$; then the $\tilde{d}$-dimensional Euclidean space $E^{\tilde{d}}$ can be constructed as the direct product $I\times\ldots\tilde{d}\ldots\times I$ of $\tilde{d}$ identical representations of the interval I [21,22,24]. Note that any coordinate system in $E^{\tilde{d}}$ clearly would contain $\tilde{d}$ independent coordinates $x_1,\ldots,x_{\tilde{d}}$, each defined on a set topologically equivalent to I, with the Hausdorff dimension equal to one. This consideration suggests the following formal definition of the space $E^{\tilde{d}}$, where $\tilde{d}=2D/(2+\theta)$ is fractional. Consider a continuous self-affine fractal curve whose Hausdorff dimension is chosen to be $(2+\theta)/2 \geqslant 1$ and let I_θ denote the "unit" element of this curve, so that I_θ is reduced to I when $\theta \equiv 0$. (The concept of the self-affine fractal curve is discussed in detail in, e.g., Refs. [1,7].) We now *define* the space $E^{\tilde{d}}$ ($\tilde{d}$ is the fractional) as the direct product $I_\theta \times \ldots \tilde{d} \ldots \times I_\theta$ of $\tilde{d}$ identical representations of the element I_θ. This definition implies that any coordinate system in $E^{\tilde{d}}$ ($\tilde{d}$ is the fractional) formally contains the *fractional* number $\tilde{d}$ of *independent* coordinates $x_1, \ldots, x_{\tilde{d}}$, each defined on a set topologically equivalent to I_θ, with the Hausdorff dimension equal to $(2+\theta)/2$. (Note that this dimension exceeds unity in general.) We also remark that the element I_θ itself can be defined as the formal direct product $I \times \ldots (2+\theta)/2 \ldots \times I$ of $(2+\theta)/2$ identical representations of the interval I. Because the dimensionality of the direct product is the algebraic sum of the dimensionalities of the terms involved, the space $E^{\tilde{d}}$ alternatively can be defined as the direct product $I \times \ldots D \ldots \times I$ of D identical unit intervals I, where we took into account that $\tilde{d}(2+\theta)/2 = D$. This alternative definition immediately shows that the volume of any domain of the space $E^{\tilde{d}}$ behaves with the given length scale χ as proportional to χ^D, in precise agreement with the geometric sense of the Hausdorff dimension D of the set F_c.

To proceed with the description of the topology of the percolating fractal set F_c, we note that the scalar random field $\psi(\mathbf{x})$ (which had been used to generate the set F_c at the threshold) is assumed to be infinite, isotropic, and statistically homogeneous. This would mean that the *global* (rather than only *local*) topology of the set F_c would be that of $\tilde{d}$-dimensional Euclidean space $E^{\tilde{d}}$, with some *fractional* value of $\tilde{d}$. We formalize this intermediate result by writing $F_c \sim E^{\tilde{d}}$, where "$\sim$" denotes the topological equivalence. (For the sake of simplicity, we ignore here the possible existence of the *isolated* fractal "voids" of all the topological dimensions between 2 and $n \geqslant 2$. Such an approximation is equivalent to saying that the set F_c is assumed to be *contractible*, i.e., could be continuously deformed into a point [22]. An inclusion of the isolated "voids" may violate the equivalence relation $F_c \sim E^{\tilde{d}}$ in general.)

Then, it is well known (see, e.g., Refs. [22,24,25]) that the Euclidean space $E^{\tilde{d}}$ ($\tilde{d}$ is a positive integer) is topologically equivalent to a $\tilde{d}$-dimensional open disk $D^{\tilde{d}}$ defined by the inequality

$$x_1^2 + \cdots + x_{\tilde{d}}^2 < 1. \tag{2}$$

In view of the above generalized definition of the space $E^{\tilde{d}}$ ($\tilde{d}$ is fractional), it is natural to assume that this topological equivalence can be extended to the fractional values of $\tilde{d}$ if one defines the *fractal* manifold $D^{\tilde{d}}$ by the same inequality (2) now formally having the *fractional* number $\tilde{d}$ of the components on the left-hand side. (The proposed topological equivalence of the manifolds $E^{\tilde{d}}$ and $D^{\tilde{d}}$ for the fractional values of $\tilde{d}$ could be rigorously considered similar to that when $\tilde{d}$ is a positive integer number [25] and will be discussed in more detail elsewhere.)

Consequently, we get $E^{\tilde{d}} \sim D^{\tilde{d}}$. Because, moreover, $F_c \sim E^{\tilde{d}}$, one concludes that the fractal manifold F_c is topologically equivalent to the $\tilde{d}$-dimensional open disk $D^{\tilde{d}}$, i.e., $F_c \sim D^{\tilde{d}}$. A ramification of this result is that the boundary ∂F_c of the manifold F_c is topologically equivalent to the boundary of the open disk $D^{\tilde{d}}$, which can be naturally defined as $(\tilde{d}-1)$-dimensional sphere $S^{\tilde{d}-1}$. (Note that for all positive integer $\tilde{d}$, the standard sphere $S^{\tilde{d}-1}$ is the boundary of the standard disk $D^{\tilde{d}}$ [25].) It is clear that the sphere $S^{\tilde{d}-1}$ with the fractional value of $\tilde{d}$ can be introduced as the fractal manifold through the equation

$$x_1^2 + \cdots + x_{\tilde{d}}^2 = 1, \tag{3}$$

where the number of the components on the left-hand side is formally fractional and equals to $\tilde{d}$.

Let us remark that the fractional number of the components on the left-hand side of expressions (2) and (3) is an explicit manifestation of the fact that the introduction of a coordinate system on a fractal object formally requires the fractional number of the independent coordinates equal to the spectral fractal dimension $\tilde{d}$. We also mention that Eq. (3) with the fractional value of the parameter $\tilde{d}$ has been used in Ref. [23] to derive a general analytical representation for the Laplacian on a fractal set, in the form originally proposed by O'Shaughnessy and Procaccia [8] for the diffusion equation on fractals.

With use of Eq. (3), it is straightforward to obtain that the sphere $S^{\tilde{d}-1}$ is seen from its center under the solid angle

$$\Omega_{\tilde{d}} = \tilde{d} \frac{\pi^{\tilde{d}/2}}{\Gamma(\tilde{d}/2+1)}, \tag{4}$$

where Γ is the Euler gamma function. Equation (4) is an obvious extension of the well-known expression [26] for the solid angle $\Omega_{\tilde{d}}$ to the fractional values of $\tilde{d}$. From Eq. (4) one immediately recovers the familiar results $\Omega_2 = 2\pi$ for the standard circle S^1 and $\Omega_3 = 4\pi$ for the standard two-dimensional sphere S^2.

We now turn to the calculation of the solid angle $\Omega_{\tilde{d}}$ for the set F_c. More precisely, we intend to prove that the quantity $\Omega_{\tilde{d}}$ for the set F_c must be equal to the fundamental constant π in all embedding Euclidean dimensions $2 \leqslant n \leqslant 5$. Knowing the exact value of the solid angle $\Omega_{\tilde{d}}$ and making use of the explicit analytical representation (4), one then could obtain the desired value of the spectral fractal dimension $\tilde{d}$ by solving the transcendental algebraic equation $\Omega_{\tilde{d}} = \pi$.

III. PERCOLATION THRESHOLD FROM THE TOPOLOGICAL VIEWPOINT

Consider first the closed $\tilde{d}$-dimensional disk $\overline{D^{\tilde{d}}}$ that is the union of the open $\tilde{d}$-dimensional disk $D^{\tilde{d}}$ and its boundary $S^{\tilde{d}-1}$, i.e., $\overline{D^{\tilde{d}}} \sim D^{\tilde{d}} \cup S^{\tilde{d}-1}$. The closed disk $\overline{D^{\tilde{d}}}$ can be defined clearly by the inequality $x_1^2+\cdots+x_{\tilde{d}}^2\leqslant 1$ and is topologically equivalent to the closed set $\overline{F_c}\sim F_c\cup\partial F_c$, i.e., $\overline{D^{\tilde{d}}}\sim\overline{F_c}$. The deduced topological equivalence $\overline{D^{\tilde{d}}}\sim\overline{F_c}$ is the formal topological manifestation of the percolation structure of the set F_c at criticality. In particular, it immediately ensures that the set $\overline{F_c}$ includes the point at infinity P_∞ of the embedding Euclidean space E^n $(n\geqslant 2)$. (Note that inclusion of the point at infinity is the topological formalization for the divergence of the percolation correlation length at the threshold $\xi\rightarrow\infty$.) We can now identify the point P_∞ with, for instance, the northern pole N of the sphere $S^{\tilde{d}-1}$. (In the event of the positive integer $\tilde{d}$, the mapping $P_\infty\rightarrow N$ is discussed in Ref. [24].) Then, let S be the southern pole of $S^{\tilde{d}-1}$. It is clear that the two points N and S of the sphere $S^{\tilde{d}-1}$ are, in the standard sense, diametrically opposite because Eq. (3) (which formally defines the fractal manifold $S^{\tilde{d}-1}$) is invariant under the inversion of all the coordinates $x_1\rightarrow -x_1,\ldots,x_{\tilde{d}}\rightarrow -x_{\tilde{d}}$.

The important step is to notice that the manifold $\overline{D^{\tilde{d}}}$ is path connected; hence, along with the two points N and S, it also includes a path γ_ϕ that goes from N to S. Of course, the manifold $\overline{D^{\tilde{d}}}$ may include in general a variety of different paths γ_ϕ that connect the two points N and S. We are now interested in finding (if possible) a particular path $\tilde{\gamma}_{\tilde{\phi}}$ with the initial point N and the final point S, which *everywhere densely* [25] covers the manifold $\overline{D^{\tilde{d}}}$. (Roughly speaking, ''everywhere densely'' means that $\tilde{\gamma}_{\tilde{\phi}}$ passes through each point of $\overline{D^{\tilde{d}}}$. In general, there may exist such points of $\overline{D^{\tilde{d}}}$ through which $\tilde{\gamma}_{\tilde{\phi}}$ passes more than once; these would be the points of self-intersections of the curve $\tilde{\gamma}_{\tilde{\phi}}$.)

The path $\tilde{\gamma}_{\tilde{\phi}}$ may exist under the following topological constraint on the dimensionality $\tilde{d}$ of the manifold $\overline{D^{\tilde{d}}}$. In fact, because $\overline{D^{\tilde{d}}}$ is the fractal manifold, it is reasonable to consider the path $\tilde{\gamma}_{\tilde{\phi}}$ as a spatial *fractal* curve [1,7] having some Hausdorff fractal dimension $\tilde{\Delta}\geqslant 1$. This curve can *everywhere densely* cover the manifold $\overline{D^{\tilde{d}}}$ (which has been quantified as the fractal set of the Hausdorff dimension D and the spectral dimension $\tilde{d}$) only if $\tilde{\Delta}\geqslant D$. Since the Hausdorff dimension D is always not less than the spectral dimension $\tilde{d}=2D/(2+\theta)$, i.e., $D\geqslant\tilde{d}$, one must require $\tilde{\Delta}\geqslant\tilde{d}$. Taking into account that $\tilde{d}\geqslant 1$ due to the path connectedness of $\overline{D^{\tilde{d}}}$, one gets the condition

$$1\leqslant\tilde{d}\leqslant\tilde{\Delta}. \tag{5}$$

The inequality (5) shows that the fractal manifold $\overline{D^{\tilde{d}}}$ can be everywhere densely covered by the fractal curve $\tilde{\gamma}_{\tilde{\phi}}$ only for the relatively low values of the dimensionality $\tilde{d}$ that are not beyond the range (5). (In higher dimensions $\tilde{d}$, the manifold $\overline{D^{\tilde{d}}}$ would contain ''too many'' points to be densely covered by the fractal curve.)

Our attention is further concentrated on the topological properties of the curve $\tilde{\gamma}_{\tilde{\phi}}$ at the percolation *threshold*. It is intuitively clear that the concept of the threshold is associated with the smallest value possible for the parameter $\tilde{d}$, so that the manifold $\overline{D^{\tilde{d}}}$ must be ''minimal'' from the viewpoint of its dimensionality. Thus we assume that the fractal manifold $\overline{D^{\tilde{d}}}$ has the lowest possible dimensionality $\tilde{d}$ at the threshold of percolation; this lowest dimensionality must be defined from the relevant topological conditions to be discussed below. The concerned property of minimality of $\overline{D^{\tilde{d}}}$ at the threshold would then impose the corresponding restrictions on the topology of the curve $\tilde{\gamma}_{\tilde{\phi}}$, which is defined to cover the manifold $\overline{D^{\tilde{d}}}$ everywhere densely. To quantify the property of ''minimality'' in the topologically accurate way, we propose the following three-step consideration outlined in Secs. III A–III C.

A. Exclusion of points of self-intersections

Let us first recall that the mapping $\tilde{\phi}$ that generates the path $\tilde{\gamma}_{\tilde{\phi}}$ has been defined as the continuous (but not necessarily *one-to-one* and *mutually* continuous) mapping, so that self-intersections of the curve $\tilde{\gamma}_{\tilde{\phi}}$ were allowed in general (see Sec. II). On the other hand, we take into account that the dimensionality $\tilde{d}$ of the fractal manifold $\overline{D^{\tilde{d}}}$ has been assumed to take the lowest, under the relevant conditions, value at the percolation threshold. But the lower the dimensionality $\tilde{d}$ of the manifold $\overline{D^{\tilde{d}}}$, the ''shorter'' the path $\tilde{\gamma}_{\tilde{\phi}}$. (Note that this path is everywhere dense in $\overline{D^{\tilde{d}}}$.) This suggests the assumption that the property of minimality might enable one to construct an everywhere dense covering of $\overline{D^{\tilde{d}}}$ by the shortest path $\tilde{\gamma}_{\tilde{\phi}}$ possible, namely, by the fractal curve $\tilde{\gamma}_{\tilde{\phi}}$, which has *no points of self-intersections*. In other words, the shortest path $\tilde{\gamma}_{\tilde{\phi}}$ yielding, by definition, an everywhere dense covering of the manifold $\overline{D^{\tilde{d}}}$ would be the one that passes through each point of $\overline{D^{\tilde{d}}}$ *once and only once*. Thus the condition that $\tilde{\gamma}_{\tilde{\phi}}$ has no points of self-intersections would be that topological restriction on the geometry of $\tilde{\gamma}_{\tilde{\phi}}$ that we associate with the property of minimality of $\overline{D^{\tilde{d}}}$ at the percolation threshold. We use this restriction in Sec. IV to calculate the desired numerical value of the dimensionality $\tilde{d}$ that would appear in the modified form (1) of the AO conjecture.

Having required that the fractal manifold $\overline{D^{\tilde{d}}}$ can be everywhere densely covered by the path $\tilde{\gamma}_{\tilde{\phi}}$ that has no points of self-intersections, we immediately impose an essential topological constraint on the admissible values of the dimensionality $\tilde{d}$. In fact, it is clear that such a particular covering cannot be constructed for the sufficiently high values of $\tilde{d}$; for instance, this is already impossible for $\tilde{d}=2$. The relevant example is the well-known Peano curve [5,27], which is defined as the plane path densely covering the two-dimensional unit square $\overline{I}\times\overline{I}$. This curve, however, has an infinite number of points of self-intersections (for instance, of multiplicity 4 [27]) where different segments of the curve are in contact with each other. Consequently, the condition that $\tilde{\gamma}_{\tilde{\phi}}$ has *no* points of self-intersections would already imply that $\tilde{d}$ is *less* than 2, i.e., $1\leqslant\tilde{d}<2$. A more precise upper bound for $\tilde{d}$ in this inequality will be obtained below (see Sec. III B).

The imposed condition for the path $\tilde{\gamma}_{\tilde{\phi}}$ to have no points of self-intersections means that the corresponding mapping $\tilde{\phi}$: $\overline{I} \to \overline{D^{\tilde{d}}}$ is a *one-to-one continuous* mapping rather than simply continuous. ''One-to-one'' indicates that the points of the unit interval $\overline{I}$ are in a one-to-one correspondence with the points of the manifold $\overline{D^{\tilde{d}}}$ (up to a set of negligible measure), so that the inverse mapping $\tilde{\phi}^{-1}$: $\overline{D^{\tilde{d}}} \to \overline{I}$ can be introduced. We now prove that for such $\tilde{\phi}$ the inverse mapping $\tilde{\phi}^{-1}$ is also continuous, so that the path $\tilde{\gamma}_{\tilde{\phi}}$ is actually generated by a *one-to-one mutually continuous* mapping, $\tilde{\phi}$.

Indeed, consider a one-to-one continuous mapping $\tilde{\phi}$ of the unit interval $\overline{I}$ on $\overline{D^{\tilde{d}}}$, i.e., $\tilde{\phi}$: $\overline{I} \to \overline{D^{\tilde{d}}}$. Because $\tilde{\phi}$ is one to one, we can introduce the *order relation* for the points of $\overline{D^{\tilde{d}}}$ in the following way. Define the parameter $t \in \overline{I}$ that ranges over the unit interval $\overline{I}$. Then the mapping $\tilde{\phi}$ can be given by a continuous single-valued function of the variable t, i.e., $\tilde{\phi} = \tilde{\phi}(t)$, where the points $\tilde{\phi}(t)$ range over $\overline{D^{\tilde{d}}}$ for t varying between 0 and 1. We now introduce the order relation on $\overline{D^{\tilde{d}}}$ by setting that $\tilde{\phi}(t_1) > \tilde{\phi}(t_2)$ if and only if $t_1 > t_2$ for all $t_1, t_2 \in \overline{I}$. This order relation makes the function $\tilde{\phi}(t)$ monotonically increasing on $\overline{I}$. [One could define, alternatively, that $\tilde{\phi}(t_1) < \tilde{\phi}(t_2)$ if and only if $t_1 > t_2$ for all $t_1, t_2 \in \overline{I}$, making the function $\tilde{\phi}(t)$ monotonically decreasing on the interval $\overline{I}$.] Thus the function $\tilde{\phi}(t)$ becomes a single valued continuous *monotonic* function on the interval $\overline{I}$, with the points $\tilde{\phi}(t)$ ranging over $\overline{D^{\tilde{d}}}$. It is widely known from the standard analysis (see, e.g., Ref. [28]) that for such a function $\tilde{\phi}(t)$ with the above *order relation*, the inverse function $\tilde{\phi}^{-1}$ is well defined, single-valued, and continuous. Q.E.D.

Our result, therefore, says that the property of minimality of $\overline{D^{\tilde{d}}}$ makes it possible to construct an everywhere dense covering of $\overline{D^{\tilde{d}}}$ by the specific path $\tilde{\gamma}_{\tilde{\phi}}$ where the mapping $\tilde{\phi}$: $\overline{I} \to \overline{D^{\tilde{d}}}$ is one to one and mutually continuous (up to a set of negligible measure) rather than simply continuous. Moreover, the very existence of such a path already implies that the dimensionality $\tilde{d}$ of the manifold $\overline{D^{\tilde{d}}}$ is less than 2, i.e., $1 \leqslant \tilde{d} < 2$. A one-to-one mutually continuous mapping is usually termed ''homeomorphism'' [22,25,27]; thus we can formulate our conclusion by saying that the property of minimality of $\overline{D^{\tilde{d}}}$ implies the existence of the path $\tilde{\gamma}_{\tilde{\phi}}$ that everywhere densely covers the manifold $\overline{D^{\tilde{d}}}$ and is homeomorphic to the unit interval $\overline{I}$. (Note that in the framework of the present study, we consider the term ''homeomorphism'' as a weaker topological concept than the ''topological equivalence.'' In fact, we imply that the topological equivalence of two fractal manifolds preserves their fractal geometry, so that any two topologically equivalent fractal manifolds necessarily have the same geometric characteristics. Meanwhile, two homeomorphic manifolds may have different fractal structure. A well-known example is the construction of the Koch curve from the unit interval $\overline{I}$ [5,10,18]. Such a construction provides a homeomorphism between the Koch curve and the unit interval $\overline{I}$; however, the Koch curve is the fractal object of the Hausdorff dimension log4/log3≈1.26 . . . >1 [5,10,18], whereas $\overline{I}$ is a segment of a smooth curve whose Hausdorff dimension is clearly equal to one.)

B. Constraint on the dimensionality $\tilde{d}$

We now strengthen the above inequality $1 \leqslant \tilde{d} < 2$ by considering the topological consequences of the conclusion that the path $\tilde{\gamma}_{\tilde{\phi}}$ is homeomorphic to $\overline{I}$. Indeed, there is a remarkable topological result [27] saying that the curve $\tilde{\gamma}_{\tilde{\phi}}$ that is homeomorphic to the interval $\overline{I}$ can be embedded (without self-intersections) into the specific topological object known as the Sierpinski carpet. (The Sierpinski carpet therefore can be treated as the *universal covering* for the curve $\tilde{\gamma}_{\tilde{\phi}}$ homeomorphic to $\overline{I}$ [27].) The topology of the Sierpinski carpet had been widely investigated; it can be shown (see, e.g., Ref. [18]) that the Sierpinski carpet is the plane fractal set and that its Hausdorff fractal dimension is equal to log8/log3≈1.89 . . . . Because the property for the curve $\tilde{\gamma}_{\tilde{\phi}}$ to be homeomorphic to $\overline{I}$ is the topological invariant [27], one should require that the Hausdorff dimension $\tilde{\Delta}$ of the curve $\tilde{\gamma}_{\tilde{\phi}}$ should not exceed the value log8/log3 for its universal covering, i.e., $\tilde{\Delta} \leqslant \log 8/\log 3$. From the inequality (5) one then gets the constraint on the dimensionality $\tilde{d}$ in the form

$$1 \leqslant \tilde{d} \leqslant \log 8/\log 3 \approx 1.89 \ldots < 2, \tag{6}$$

providing the *existence* of the everywhere dense covering of the manifold $\overline{D^{\tilde{d}}}$ by the fractal curve $\tilde{\gamma}_{\tilde{\phi}}$, which has no points of self-intersections.

C. Definition of the critical solid angle

We are now ready to quantify the property of minimality of the manifold $\overline{D^{\tilde{d}}}$ in the final form. Indeed, we argued above that the minimal fractal manifold $\overline{D^{\tilde{d}}}$ can be everywhere densely covered by the fractal curve $\tilde{\gamma}_{\tilde{\phi}}$ that is homeomorphic to the unit interval $\overline{I}$. Assume that this curve is seen from the center of the manifold $\overline{D^{\tilde{d}}}$ (i.e., from the center of the sphere $S^{\tilde{d}-1}$) under the solid angle $\tilde{\Omega}$. Because $\tilde{\gamma}_{\tilde{\phi}}$ is everywhere dense in $\overline{D^{\tilde{d}}}$, the solid angle $\tilde{\Omega}$ is easily seen to coincide with the solid angle $\Omega_{\tilde{d}}$ that defines the sphere $S^{\tilde{d}-1}$ [see Eq. (4)], i.e., $\Omega_{\tilde{d}} = \tilde{\Omega}$.

Then, the concept of the percolation threshold has been associated with the smallest value possible for the parameter $\tilde{d}$, yielding the critical dimensionality of the fractal manifold $\overline{D^{\tilde{d}}}$. [This critical dimensionality $\tilde{d}$ must be implied in the modified form (1) of the AO conjecture.] Taking into account that the solid angle $\Omega_{\tilde{d}}$ is the monotonically increasing analytical function of the variable $\tilde{d}$ [in, at least, the range

(6) that is of interest], one then defines the corresponding critical value of the solid angle $\Omega_{\tilde{d},min}$ to be assigned to the threshold of percolation. Thus the critical solid angle $\Omega_{\tilde{d},min}$ is given by Eq. (4) when substituting the critical value of the dimensionality $\tilde{d}$. In view of the above, $\Omega_{\tilde{d},min}$ must coincide with the smallest value possible for the solid angle $\tilde{\Omega}$ (this being the solid angle determined by the fractal curve $\tilde{\gamma}_{\tilde{\phi}}$), which we denote by $\tilde{\Omega}_{min}$. Hence, at the threshold of percolation

$$\Omega_{\tilde{d}} = \Omega_{\tilde{d},min} = \tilde{\Omega}_{min}. \tag{7}$$

We now turn to the explicit calculation of the quantity $\tilde{\Omega}_{min}$. This would give us the key to obtain the desired value of the dimensionality $\tilde{d}$ at the percolation threshold by combining Eqs. (4) and (7).

IV. CALCULATION OF THE CRITICAL PARAMETERS

To obtain the numerical value of the solid angle $\tilde{\Omega}_{min}$, we are reminded that the initial and the final points of the path $\tilde{\gamma}_{\tilde{\phi}}$ are, respectively, the two diametrically opposite poles of the sphere $S^{\tilde{d}-1}$, i.e., the points N and S. It is clear that the path $\tilde{\gamma}_{\tilde{\phi}}$ going from N to S cannot be based on the solid angle $\tilde{\Omega}$ smaller than the one that defines the *standard arch of the semicircle* with the end points N and S. [Note that the inequality (6) provides the existence of a homeomorphism between this arch and $\tilde{\gamma}_{\tilde{\phi}}$.] Assuming the embedding dimensionality n to be greater than one, i.e., $n \geqslant 2$, one immediately concludes that the standard arch of the semicircle with the diametrically opposite end points N and S is based on the solid angle $\Omega_2/2 = \pi$, where $\Omega_2 = 2\pi$ is the total solid angle for the standard circle S^1 [see Eq. (4)]. Hence $\tilde{\Omega} \geqslant \Omega_2/2 = \pi$ and, consequently,

$$\tilde{\Omega}_{min} = \pi. \tag{8}$$

Because, moreover, $\Omega_{\tilde{d},min} = \tilde{\Omega}_{min}$ at criticality [see Eq. (7)], one finally gets the critical value $\Omega_{\tilde{d},min}$ of the solid angle $\Omega_{\tilde{d}}$, i.e.,

$$\Omega_{\tilde{d},min} = \pi. \tag{9}$$

Equation (9) proves the above assertion that the solid angle $\Omega_{\tilde{d}}$ for the percolating fractal set $\overline{F_c} \sim \overline{D^{\tilde{d}}}$ is equal to the fundamental constant π at the threshold of percolation. The validity of this result assumes, at least, that the topological dimension of the embedding Euclidean space n is greater than one, i.e., $n \geqslant 2$. In Sec. V, we prove that n has the upper bound equal to 5, so that Eqs. (7)–(9) are actually valid for $2 \leqslant n \leqslant 5$.

Using Eq. (9), it is now straightforward to obtain the spectral fractal dimension $\tilde{d}$ at the percolation threshold. In fact, taking into account that the solid angle $\Omega_{\tilde{d}}$ is the function of the spectral fractal dimension $\tilde{d}$ and combining Eqs. (4) and (9), we find that the parameter $\tilde{d}$ at the percolation threshold obeys the transcendental algebraic equation of completely topological origin,

$$\tilde{d}\frac{\pi^{\tilde{d}/2}}{\Gamma(\tilde{d}/2+1)} = \pi, \tag{10}$$

which holds for $2 \leqslant n \leqslant 5$ (see Sec. V). Calculation of the spectral fractal dimension $\tilde{d}$ from Eq. (10) is immediate. Indeed, under the restriction (6), the solution to Eq. (10) is easily proven to be *unique*. From a simple numerical consideration one finally obtains

$$\tilde{d} = 1.327 \pm 0.001. \tag{11}$$

The result (11) yields the desired, critical value of the spectral fractal dimension $\tilde{d}$ for the percolating sets at the threshold of percolation [see Eq. (1)]. The constant on the right-hand side of Eq. (11) is remarkably close to (although slightly *smaller* than) the original AO result 4/3 and has the fundamental topological nature, having been deduced from the most general geometric properties of the fractal manifolds. These properties have been described above by such fundamental topological concepts as path connectedness, topological equivalence, and everywhere dense covering and are manifest in expressions (2)–(9).

The validity of the result (11) assumes that the fractal manifold $\overline{D^{\tilde{d}}}$ (which has been introduced to approximate the topology of the percolating fractal set F_c at criticality) could be everywhere densely covered by the fractal curve $\tilde{\gamma}_{\tilde{\phi}}$ without points of self-intersections. The necessary constraints on the *intrinsic* geometry of the manifold $\overline{D^{\tilde{d}}}$ related to the existence of such a particular covering already have been discussed above [see, e.g., the inequality (6)]. However, the important issue of *embedding* of the manifold $\overline{D^{\tilde{d}}}$ into n-dimensional Euclidean space E^n $(n \geqslant 2)$ and the relevant constraints on the parameter n must be also pointed out. We prove below that the fractal manifold $\overline{D^{\tilde{d}}}$ whose dimensionality $\tilde{d}$ obeys the inequality (6) can be embedded into a not more than five-dimensional Euclidean space E^n, i.e., $2 \leqslant n \leqslant 5$. The derivation of this inequality will complete the topological proof of the AO conjecture in the form (1). We proceed as follows.

V. CONSTRAINT ON THE EMBEDDING DIMENSION

Let $0 < \varepsilon \ll 1$ be an arbitrary small positive parameter and let A_i be an arbitrary point of the curve $\tilde{\gamma}_{\tilde{\phi}}$, i.e., $A_i \in \tilde{\gamma}_{\tilde{\phi}}$. Then, let n denote the topological dimension of the embedding Euclidean space E^n; we assume that n is an integer number greater than one, i.e., $n \geqslant 2$.

Surround the point A_i with the closed n-dimensional disk $\overline{D_i^n}(\varepsilon)$ of radius ε, so that A_i would be the center of $\overline{D_i^n}(\varepsilon)$. The disk $\overline{D_i^n}(\varepsilon)$ could be regarded clearly as the ε vicinity of the point A_i in the embedding Euclidean space

E^n. The boundary of the disk $\overline{D_i^n}(\varepsilon)$ is defined as the $(n-1)$-dimensional sphere $S_i^{n-1}(\varepsilon)$, i.e., $\partial\overline{D_i^n}(\varepsilon)=S_i^{n-1}(\varepsilon)$.

Consider the intersection $\overline{D_i^n}(\varepsilon)\cap\overline{D^{\tilde{d}}}$ of the disk $\overline{D_i^n}(\varepsilon)$ with the fractal manifold $\overline{D^{\tilde{d}}}$. In the most general case, the set of points $\overline{D_i^n}(\varepsilon)\cap\overline{D^{\tilde{d}}}$ is disconnected since it may include more than one ''piece'' of $\overline{D^{\tilde{d}}}$. We are now interested in the *path-connected* subset of $\overline{D_i^n}(\varepsilon)\cap\overline{D^{\tilde{d}}}$ that includes the point A_i. Denote this subset by $\alpha_i(\varepsilon)$, i.e., $\alpha_i(\varepsilon)\subset\overline{D_i^n}(\varepsilon)\cap\overline{D^{\tilde{d}}}$, and $A_i\in\alpha_i(\varepsilon)$.

Then, let $\tilde{\gamma}_i(\varepsilon)$ be an element of the path $\tilde{\gamma}_{\tilde{\phi}}$ that everywhere densely covers the subset $\alpha_i(\varepsilon)$. Assuming the inequality (6), one concludes that the element $\tilde{\gamma}_i(\varepsilon)$ can be embedded (without self-intersections) into an element of the Euclidean plane E^2, which we denote by $E_i^2(\varepsilon)$ (see Sec. III). Hence there exists a one-to-one mutually continuous mapping $\tilde{\psi}_i$ of the element $\tilde{\gamma}_i(\varepsilon)$ into $E_i^2(\varepsilon)$, i.e., $\tilde{\psi}_i:\ \tilde{\gamma}_i(\varepsilon)\rightarrow E_i^2(\varepsilon)$.

Since A_i was assumed to be *arbitrary* point of the curve $\tilde{\gamma}_{\tilde{\phi}}$, one can construct a sequence of the mutually intersecting disks $\overline{D_i^n}(\varepsilon)$ that covers the whole $\tilde{\gamma}_{\tilde{\phi}}$ (and hence the whole $\overline{D^{\tilde{d}}}$) for the index i varying, say, between 1 and some integer number $N(\varepsilon)\geqslant 1$. Indeed, one could find, for instance, the two end points of the element $\tilde{\gamma}_i(\varepsilon)$, which lie on the sphere $S_i^{n-1}(\varepsilon)$ and we denote by A_{i-1} and A_{i+1}. Then, one defines the disks $\overline{D_{i-1}^n}(\varepsilon)$ and $\overline{D_{i+1}^n}(\varepsilon)$ centered at the points A_{i-1} and A_{i+1}, respectively. Similar to the above, one finds the new elements $\tilde{\gamma}_{i-1}(\varepsilon)$ and $\tilde{\gamma}_{i+1}(\varepsilon)$ of the curve $\tilde{\gamma}_{\tilde{\phi}}$; these two elements are easily seen to have the common end point A_i, which is the center of the disk $\overline{D_i^n}(\varepsilon)$. In addition, each of the elements $\tilde{\gamma}_{i-1}(\varepsilon)$ and $\tilde{\gamma}_{i+1}(\varepsilon)$ has one more end point opposite to A_i. Let these end points be A_{i-2} for $\tilde{\gamma}_{i-1}(\varepsilon)$ and A_{i+2} for $\tilde{\gamma}_{i+1}(\varepsilon)$. The points A_{i-2} and A_{i+2} are then identified with the centers of the new disks $\overline{D_{i-2}^n}(\varepsilon)$ and $\overline{D_{i+2}^n}(\varepsilon)$ and so on until one reaches the northern N and the southern S poles of the sphere $S^{\tilde{d}-1}$. (Note that the points N and S are, by definition, the end points of the curve $\tilde{\gamma}_{\tilde{\phi}}$.) For the positive (nonzero) values of ε, the total number, $N(\varepsilon)$ of disks $\overline{D_i^n}(\varepsilon)$ needed to cover the whole $\tilde{\gamma}_{\tilde{\phi}}$, is *finite*, i.e., $N(\varepsilon)<\infty$ for $\varepsilon>0$. Hence all the disks $\overline{D_i^n}(\varepsilon)$ covering the curve $\tilde{\gamma}_{\tilde{\phi}}$ can be enumerated by the index i varying between 1 and $N(\varepsilon)$, i.e., $1\leqslant i\leqslant N(\varepsilon)$.

The next step is to define the mapping $\tilde{\psi}_i:\ \tilde{\gamma}_i(\varepsilon)\rightarrow E_i^2(\varepsilon)$ for all $1\leqslant i\leqslant N(\varepsilon)$. This generates the sequence of the mutually intersecting elements $E_i^2(\varepsilon)$, which lies, by assumption, in the Euclidean space E^n with some $n\geqslant 2$. Let $M^2=\cup_i E_i^2(\varepsilon)$ be the union of all the elements $E_i^2(\varepsilon)$ for $1\leqslant i\leqslant N(\varepsilon)$. We now claim that M^2 has the topology of the compact two-dimensional manifold embedded into E^n $(n\geqslant 2)$. (For the accurate definition of the term ''compact'' see, e.g., Refs. [24,25].) Indeed, M^2 is explicitly represented as the union of the finite number $N(\varepsilon)$ of the elements $E_i^2(\varepsilon)$, each topologically equivalent to a domain of the two-dimensional Euclidean plane E^2. The transition from one local domain $E_i^2(\varepsilon)$ to another [e.g., to $E_{i\pm 1}^2(\varepsilon)$] is unambiguously defined by the given sequence of the mutually intersecting disks $\overline{D_i^n}(\varepsilon)$ for $1\leqslant i\leqslant N(\varepsilon)$ and might be quantified in the explicit form by the $N(\varepsilon)\times N(\varepsilon)$ matrix of the corresponding transition functions. (The matrix of the transition functions could be obtained directly following, e.g., Refs. [21,24]. See also the relevant discussion in Sec. II.) After all, the compactness of M^2 [21,24,25] immediately follows from the fact that the number $N(\varepsilon)$ of all the domains $E_i^2(\varepsilon)$ is *finite* for $\varepsilon>0$.

It is theoretically important to note that the dimensionality of the manifold $M^2=\cup_i E_i^2(\varepsilon)$ is defined by the dimensionality of the *local* elements $E_i^2(\varepsilon)$ and is equal to 2 as soon as each $E_i^2(\varepsilon)$ is topologically equivalent to a domain of the two-dimensional Euclidean plane E^2. The latter, of course, means that each *local* element $E_i^2(\varepsilon)$ can be embedded into E^2. This property, however, is only *local* because the *global* topology of the (entire) manifold $M^2=\cup_i E_i^2(\varepsilon)$ may need more embedding dimensions n, depending on how the local elements $E_i^2(\varepsilon)$ are glued together to form the global geometric structure of M^2. Hence the embedding Euclidean space for a two-dimensional manifold M^2 is not E^2 in general, but could be more dimensional. A relevant example might be the *Möbius band* which is a two-dimensional compact manifold but cannot be embedded into E^2 [21]. Thus we are already prepared to conclude that the embedding dimension n for the (compact) manifold M^s of the dimensionality s might be greater in general than s, i.e., $n\geqslant s$.

The *global* embedding of manifolds into the Euclidean space E^n is quantified by the famous *weak Whitney theorem* (see, e.g., Ref. [21]). The conclusion of this theorem is that (i) any compact manifold M^s whose dimensionality s is integer can be *always* embedded into the Euclidean space E^n of the sufficiently high dimensionality $n\geqslant s$; moreover, (ii) the value of n does not exceed the number $2s+1$ (for the commonly implied *prime* embedding that uses *all* dimensions n), so that n could be found in the range $s\leqslant n\leqslant 2s+1$. (For the full discussion of the particular assumptions that might be imposed on the topology of the manifold M^s see, e.g., Ref. [21].) In other words, the Whitney theorem *guarantees* that the compact manifold M^s can be embedded into E^{2s+1}. This, of course, does not exclude that M^s could be already embedded into the Euclidean space E^n whose dimensionality n is *smaller* than $2s+1$ if the appropriate topological conditions hold. Hence the value of $n=2s+1$ must be treated as the *upper* prime embedding dimension for M^s; it is clear that any embedding of M^s into the Euclidean space E^m of the dimensionality $m>2s+1$ could be reduced, according to the Whitney theorem, to the prime embedding into some E^n with $n\leqslant 2s+1$.

We are now going back to the manifold $M^2=\cup_i E_i^2(\varepsilon)$. For this manifold, the above parameter s is equal to 2, i.e., $s=2$. Making use of the Whitney theorem, one immediately concludes that the prime embedding of M^2 could be found into at most five-dimensional Euclidean space E^5, where we took into account that $2s+1=5$ for $s=2$. In general, the dimensionality n of the embedding Euclidean space E^n would be not less than the dimensionality of the manifold M^2 itself, i.e., $n\geqslant 2$, and not more than the dimensionality $2s+1=5$ guaranteed by the Whitney theorem, i.e., $n\leqslant 5$. Finally, one gets

$$2 \leqslant n \leqslant 2s+1=5, \tag{12}$$

yielding the desired constraint on the dimensionality n. The result (12) completes the proof of the AO conjecture in the form (1). Q.E.D.

VI. SUMMARY AND CONCLUSIONS

In summary, we have shown that the Alexander-Orbach conjecture [3], which assigns the universal (mean-field) value 4/3 to the spectral fractal dimension $\tilde{d}$ at the percolation threshold in all embedding Euclidean dimensions $n \geqslant 2$, might be improved for $2 \leqslant n \leqslant 5$ to have the modified form given by Eq. (1). The improved form (1) of this conjecture would then imply the value approximately equal to 1.327 for the spectral dimension $\tilde{d}$ at the percolation threshold in all embedding dimensions $2 \leqslant n \leqslant 5$. We note that this improved value is slightly *smaller* than the original AO result 4/3 (which, as we argued above, remains valid for $n \geqslant 6$).

To advocate the improved form (1) of the AO conjecture, we have proposed an unconventional analytical approach based on the most general methods of the differential topology. We have shown that the basic topological features of the percolating fractal sets might be effectively investigated with the introduction of the concept of the *fractal manifold*, which extends the widely known concept of the smooth k-dimensional manifold (k is a positive integer) to the fractional values of the dimensionality k. In the framework of the present study, we associated the topology of percolation at the threshold with that of the fractal manifold $\overline{D^{\tilde{d}}}$. This manifold was defined as the closed $\tilde{d}$-dimensional disk, where the parameter $\tilde{d}$ coincides with the spectral fractal dimension of the percolating set.

The introduction of the fractal manifold $\overline{D^{\tilde{d}}}$ has made it possible to find a simple topological condition that identifies the threshold of percolation. In fact, we argued that the boundary of the manifold $\overline{D^{\tilde{d}}}$ [which is defined as $(\tilde{d}-1)$-dimensional sphere $S^{\tilde{d}-1}$] must be seen from inside under the universal solid angle $\Omega_{\tilde{d}}=\pi$ just at the percolation threshold [see Eqs. (8) and (9)]. With use of the Whitney theorem [21], we have shown that the result $\Omega_{\tilde{d}}=\pi$ is valid for percolation in all embedding Euclidean dimensions $2 \leqslant n \leqslant 5$.

In view of Eq. (4), the proposed condition $\Omega_{\tilde{d}}=\pi$ leads to the transcendental algebraic equation $\tilde{d}\pi^{\tilde{d}/2}/\Gamma(\tilde{d}/2+1)=\pi$ for the spectral fractal dimension $\tilde{d}$ at criticality, which holds for $2 \leqslant n \leqslant 5$. Under the relevant topological constraints quantified by the inequality (6), the solution to this equation is unique, yielding the value of the spectral fractal dimension at the percolation threshold $\tilde{d}=1.327 \pm 0.001$ for all $2 \leqslant n \leqslant 5$ [see Eq. (1)]. This result has the fundamental topological origin and is related to such basic topological concepts as path connectedness, topological equivalence, and everywhere dense covering. This proves the AO conjecture in the modified form (1).

It must be emphasized that the analytical approach proposed in the present study assumes that the percolating fractal sets at the threshold have the topological property of *contractibility* [22]. Such an approach actually ignores the possible role of the isolated voids of the topological dimensions between 2 and $n \geqslant 2$, where n is the embedding Euclidean dimension (see Sec. II). Thus the result (11) might be exact only for the contractible percolating fractal sets.

A comprehensive investigation of the *noncontractible* fractal manifolds that include the isolated voids might be the topological problem of the outstanding significance. For instance, this problem might be associated with the problem of the *topological classification* of the fractal manifolds from the viewpoint of algebraic codes when each code identifies the topological type of the fractal manifold through some classification algorithm [29]. A computer realization of such classification codes might then answer (in the algebraic way) the important, for the actual applications, question whether or not the two given algebraic codes correspond to the *topologically equivalent* fractal manifolds. (This might be essential also for the direct numerical recognition of the topological equivalence of the fractal manifolds of given dimensionality.) A similar problem of the topological classification of the *smooth* manifolds has received a good deal of attention in the modern topology and the exact algorithmic classification of the two-dimensional smooth manifolds has been developed [29]. (It is worth mentioning, however, that the algorithmic classification of the three-dimensional smooth manifolds meets already considerable difficulties, whereas the classification of the smooth manifolds whose dimensionality is not less than 4 cannot be performed in principle [29].)

In spite of the considerable difficulties that might arise when analyzing the topological properties of the noncontractible fractal manifolds, however, we might attempt to anticipate the possible values of the spectral fractal dimension $\tilde{d}$ at the threshold of percolation for the noncontractible percolating sets. In fact, it is intuitively clear that the inclusion of voids would act towards a more intensified percolation since the convergence of the percolating set to infinity would be quicker in this case. Hence the percolation threshold then might be achieved for the *lower* value of the spectral fractal dimension $\tilde{d}$ compared to the value (11) for the contractible sets. Thus, one could expect that the fractal dimension $\tilde{d}$ might be slightly *smaller* than approximately 1.327 for the most general topology of percolation when the voids might be present in a considerable range of length scales. Such a conclusion might be supported, for instance, by the numerical results of Normand *et al.* [30], which are among the most accurate; these results were obtained for the plane percolation $(n=2)$, yielding $\tilde{d} \approx 1.321$, which is indeed slightly smaller than the value (11). A more detailed consideration of the topological properties of the noncontractible percolating fractal sets might be an attractive point for future studies.

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